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ON A LIAPUNOV FUNCTION IN THE PROBLEM OF MOTION OF A RIGID BODY

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We examine the application of a Liapunov function in the form of a quadratic form with coefficients which are time functions, to investigate the stability of the permanent rotations of a rigid body with one point fastened on a moving base. When investigating the stability of the permanent rotations of a rigid body with one fixed point in a potential force field, as the Liapunov function we select, as a rule, the bundle of integrals of the equations of perturbed motion, starting with terms of second order relative to the perturbations (for example, see [1]). The derivative of this function relative to the equations of perturbed motion equals zero, therefore, the positive definiteness of the quadratic form of the function indicated ensures the stability of the unperturbed motion. However, if the fixed point of the rigid body is located on a moving base (performing a specified motion), then the construction of the Liapunov function in the form of a bundle of first integrals of the equations of motion is impossible because of the absence of the energy integral. Therefore, it is necessary to try to find other means of constructing the Liapunov function; we examine below one of the methods for such a construction.

1. We consider the function

$$V(t, x_1, x_2, \ldots, x_n) = \Sigma a_{ij}(t) x_i x_j \tag{1.1}$$

given in the region

$$t \geqslant t_0 > 0, \ |x_s| < h \ (s = 1, 2, \ldots, n) \tag{1.2}$$

where t_0 and h are constants. Let the coefficients of quadratic form (1.1) be

$$a_{ij}(t) = a_{ji}(t) = f(t) c_{ij}$$

Here c_{ij} are constant system parameters and $f(t)$ is a positive periodic time function with period $\varepsilon < 1/2$, admitting of discontinuities of the first kind at points $t_m = t_0 + m\varepsilon$ $(m = 1, 2, \ldots)$. In the interval $t_{m-1} < t < t_m$ the graph of function $f(t)$ is a straight line parallel to the bisector of the second and fourth quadrants. It is obvious that the function $f(t)$ introduced and the coefficients of quadratic form (1.1) possess, everywhere except at the points t_m, the properties

$$1 - \varepsilon < f(t) < 1, \ f^{\cdot}(t) = 1$$

$$(1 - \varepsilon) |c_{ij}| \leqslant |a_{ij}(t)| \leqslant c_{ij}, \ a_{ij}^{\cdot}(t) = -c_{ij}$$

Function (1.1) vanishes at the origin of the space of $x_1, x_2, \ldots, x_n$ and takes only positive values in a neighborhood of the origin if there exists a positive-definite quadratic form with constant coefficients

$$W(x_1, x_2, \ldots, x_n) = (1 - 2\varepsilon) \Sigma c_{ij} x_i x_j$$

i.e. when the inequalities

$$C_s = \begin{vmatrix} c_{11} \dots c_{1s} \\ c_{s1} \dots c_{ss} \end{vmatrix} > 0 \qquad (s = 1, 2, \dots, n) \tag{1.3}$$

are fulfilled. Indeed, in this case the function

$$V - W = \sum b_{ij}(t) x_i x_j, \qquad b_{ij}(t) = c_{ij}[f(t) - (1 - 2\varepsilon)]$$

is a quadratic form all of whose diagonal minors

$$B_s = [f(t) - (1 - 2\varepsilon)]^s C_s \tag{1.4}$$

are positive time functions for all $t \geqslant t_0 > 0$ since the time function within the brackets in expression (1.4) is contained between ε and 2ε and, consequently, is a positive function, while on the basis of conditions (1.3) the constants $C_s > 0$. Thus, the relation

$$V(t, x_1, x_2, \dots, x_n) > W(x_1, x_2, \dots, x_n) > 0 \tag{1.5}$$

holds under conditions (1.3), i.e. in region (1.2) the form (1.1) is a positive-definite function depending on t.

By arguing analogously we can show that function V satisfies the condition $V < W_1$, where W_1 is a positive-definite quadratic form with constant coefficients. Therefore, we can conclude that the one-parameter family of cycles $V = c > 0$ in the space of variables x_s is contained between two constant cycles $W = c$ and $W_1 = c$.

Let us now consider the differential equations of a perturbed motion, of the form

$$x_s^{\cdot} = X_s(t, x_1, x_2, \dots, x_n)$$

Let the total time derivative of function V, taken relative to these equations, i.e. the expression

$$V^{\cdot} = \partial V / \partial t + \Sigma X_s \partial V / \partial x_s$$

existing for all t except the points of discontinuity of function $f(t)$ and predetermined at these points, be a negative function or be identically zero. Then the unperturbed motion is Liapunov-stable, i.e. the trajectory of the motion of the representative point starting from the positions $\Sigma x_{s0}^2 = \Sigma x_s^2(t_0) \leqslant \lambda$, does not go outside of the sphere $\Sigma x_s^2 = \delta$, where δ is a positive number, $\lambda = \lambda(\delta)$. As a matter of fact it is obvious that on any sphere $\Sigma x_s^2 = \delta$ in region (1.2) of the space of variables x_s there holds the condition $W(x_1, x_2, \dots, x_n) \geqslant l$, where l is the greatest lower bound of the function W on this sphere. Then on the basis of inequality (1.5) the condition $V(t, x_1, \dots, x_n) > l$ also is fulfilled on the sphere $\Sigma x_s^2 = \delta$.

On the other hand we can also find points x_{s0} of the space of variables x_s, located in the region $\Sigma x_{s0}^2 \leqslant \lambda < \delta$, such that the condition $V(t_0, x_{10}, x_{20}, \dots, x_{n0}) < l$ is fulfilled (this is possible since $V(t_0, 0, 0, \dots, 0) = 0$). According to the condition $V^{\cdot} \leqslant 0$ we have $V(t, x_1, x_2, \dots, x_n) \leqslant V(t_0, x_{10}, x_{20}, \dots, x_{n0}) < l$, i.e. it is impossible for the representative point $(x_1, x_2, \dots, x_n)$ to hit onto the sphere $\Sigma x_s^2 = \delta$. Thus the Liapunov stability theorem is true for the function (1.1) with the stated properties and it can be taken as the Liapunov function [2].

2. With the aid of Liapunov function (1.1) we investigate the stability of the rotary motion of a Lagrange gyroscope (spinning top) with one point fastened to a moving base and located in a central Newtonian force field. Let us consider a rigid body whose principal moments of inertia are $A = B \neq C$, the center of mass is located on the Oz-axis of dynamic symmetry: $x_c = y_c = 0,\ z_c = z_0 > 0$, in a central Newtonian force field with the force functions

$$U = -mg\,(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) - \mu\,(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2)\,/\,2$$

Here x_0, y_0, z_0 are the coordinates of the center of mass in the $Oxyz$-axes directed along the principal axes of the ellipsoid of inertia of the rigid body, μ is a constant depending on the gravitational constant and on the body's distance from the attracting center , $\gamma_1, \gamma_2, \gamma_3$ are the direction cosines of the z_1-axis (connecting the attracting center with the center of the ellipsoid of inertia of the rigid body) in the $Oxyz$ coordinate system.

Suppose that the center O of the ellipsoid of inertia performs a harmonic oscillation along the z_1-axis by the law

$$z_{10} = \alpha \sin kt, \qquad \alpha, k > 0$$

Then the equations of motion of the rigid body, referred to the $Oxyz$ system, have the form

$$\begin{aligned} p^{\cdot} &= (1-\nu)\,qr + a\,(t)\,\gamma_2 - \mu\,(1-\nu)\,\gamma_2\gamma_3, & \gamma_1^{\cdot} &= r\gamma_2 - q\gamma_3 \\ q^{\cdot} &= -(1-\nu)\,pr - a\,(t)\,\gamma_1 + \mu\,(1-\nu)\,\gamma_1\gamma_3, & \gamma_2^{\cdot} &= p\gamma_3 - r\gamma_1 \\ r^{\cdot} &= 0, & \gamma_3^{\cdot} &= q\gamma_1 - p\gamma_2 \end{aligned} \tag{2.1}$$

Here p, q, r are the projections of the instanteneous angular velocity of the rigid body onto the x, y, z axes, respectively, $\nu = C\,/\,A$ is a constant, $a\,(t) = mz_0\,(g - \alpha k^2 \sin kt)\,/A$ is a known time function. Equation (2.3) admits of a particular solution

$$p = q = 0,\ r = r_0,\ \gamma_1 = \gamma_2 = 0,\ \gamma_3 = \tag{2.2}$$

which corresponds to the rotation of the rigid body with angular velocity r_0 around the axis of dynamic symmetry directed along the axis connecting the attracting center with the center of the ellipsoid of inertia.

Let us investigate the stability of the unperturbed motion (2.2) relative to the variables $p, q, r, \gamma_1, \gamma_2, \gamma_3$. In the perturbed motion we set

$$p = x_1,\ q = x_2,\ r = r_0 + x_3,\ \gamma_1 = y_1,\ \gamma_2 = y_2,\ \gamma_3 = 1 + y_3 \tag{2.3}$$

Then the equations of perturbed motion take the form

$$\begin{gathered} x_1^{\cdot} = (1-\nu)\,(r_0 + x_3)\,x_2 + a\,(t)\,y_2 - \mu\,(1-\nu)\,(1+y_3)\,y_2,\quad x_2^{\cdot} = -(1-\nu)\,(r_0 + \\ x_3)\,x_1 - a\,(t)\,y_1 + \mu\,(1-\nu)\,(1+y_3)\,y_1,\quad x_3^{\cdot} = 0,\quad y_1^{\cdot} = (r_0 + x_3)\,y_2 - x_2\,(1+y_3) \\ y_2^{\cdot} = x_1\,(1+y_3) - (r_0 + x_3)\,y_1,\ y_3^{\cdot} = x_2 y_1 - x_1 y_2 \end{gathered} \tag{2.4}$$

We obtain sufficient conditions for the stability of motion (2.2) by examining a Liapunov function of the type of form (1.1),

$$\begin{gathered} V = f\,(kt)\,\{x_1^2 - \nu r_0 x_1 y_1 + [\nu^2 r_0^2\,/\,2 - a\,(t) + \mu\,(1-\nu)]\,y_1^2 + x_2^2 - \nu r_0 x_2 y_2 + \\ [\nu^2 r_0^2\,/\,2 - a\,(t) + \mu\,(1-\nu)]\,y_2^2 + \nu^2 x_3^2 - \nu^2 r_0\,x_3 y_3 + [\nu^2 r_0^2\,/\,2 - a\,(t)]\,y_3^2\} \end{gathered} \tag{2.5}$$

In the given case, under conditions (2.6) or (2.7)

$$A > C,\ C^2 r_0^2 - 4mz_0 A\,(g + \alpha k^2) - 4\,\mu\,(A - C) > 0 \tag{2.6}$$

$$A < C,\ C^2 r_0^2 - 4mz_0 A\,(g + \alpha k^2) > 0 \tag{2.7}$$

the function (2.5) is a positive-definite quadratic form in the variables x_i, y_i, which satisfies condition (1.3). The derivative of function (2.5) by virtue of the equations of perturbed motion (2.4) is the quadratic form

$$-V^{\cdot}/k = x_1^2 - \nu r_0 x_1 y_1 + [\nu^2 r_0^2/2 - a(t) + \mu(1-\nu) + a^{\cdot}(t) f(kt)/k]\, y_1^2 + x_2^2 - \nu r_0 x_2 y_2 + [\nu^2 r_0^2/2 - a(t) + \mu(1-\nu) + a^{\cdot}(t) f(kt)/k]\, y_2^2 + \nu^2 x_3^2 - \nu^2 r_0 x_3^2 y_3 + [\nu^2 r_0^2/2 - a(t) + a^{\cdot}(t) f(kt)/k]\, y_3^2$$

which takes negative values under the conditions

$$A > C,\ C^2 r_0^2 - 4mz_0 A(g + 2\alpha k^2) - 4\mu(A - C) > 0 \qquad (2.8)$$

$$A < C,\ C^2 r_0^2 - 4mz_0 A(g + 2\alpha k^2) > 0 \qquad (2.9)$$

It is evident that when conditions (2.9) or (2.8) are fulfilled, conditions (2.6) or (2.7) are fulfilled. In other words, inequalities (2.8) or (2.9) are sufficient conditions for the stability of the unperturbed motion (2.2).

In the case $\alpha = 0$ ($k = 0$) inequality (2.9) turns into the Majewski stability condition well known in the literature [3]. The question of how close conditions (2.8) or (2.9) are to the necessary conditions for the stability of motion (2.2) is answered by an examination of the function

$$V_1 = x_1 y_2 - x_2 y_1 \qquad (2.10)$$

Indeed, in the region $V_1 > 0$ the derivative of function (2.10) by virtue of the equations of perturbed motion (2.4) has the form [3]

$$V_1^{\cdot} = x_1^{\cdot} y_2 + x_1 y_2^{\cdot} - x_2^{\cdot} y_1 - x_2 y_1^{\cdot} = x_1^2 - \nu r_0 x_1 y_1 + [a(t) - \mu(1-\nu)]\, y_1^2 + x_2^2 - \nu r_0 x_2 y_2 + [a(t) - \mu(1-\nu)]\, y_2^2$$

whose sign is the same as the sign of V_1 under the conditions

$$A < C,\ C^2 r_0^2 - 4mz_0 A(g - \alpha k^2) - 4\mu(A - C) < 0 \qquad (2.11)$$

$$A > C,\ C^2 r_0^2 - 4mz_0 A(g - \alpha k^2) < 0 \qquad (2.12)$$

Therefore, inequalities (2.11) or (2.12) are conditions for the instability of the unperturbed motion (2.2) [2].

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